

# A bound concerning primordial non-Gaussianity

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**Abstract.** Seery and Lidsey have calculated the three-point correlator of the light scalar fields, a few Hubble times after horizon exit during inflation. Lyth and Rodriguez have calculated the contribution of this correlator to the three-point correlator of the primordial curvature perturbation. We calculate an upper bound on that contribution, showing that it is too small ever to be observable.

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## 1. Introduction

The study of non-Gaussianity features in the primordial curvature perturbation  $\zeta$  has become a subject of growing interest, because they provide a valuable discriminator between different models for its origin [1, 2, 3, 4]. While the relevant scales are outside the horizon, the curvature perturbation according to the  $\delta N$  formalism is given by [5, 6] (see also [7, 8, 9, 10])

$$\zeta(\mathbf{x}, t) = \delta N(\phi_i(\mathbf{x}), \rho(t)) \quad (1)$$

$$= \sum_i N_i(t) \delta\phi_i(\mathbf{x}) + \frac{1}{2} \sum_{ij} N_{ij}(t) \delta\phi_i(\mathbf{x}) \delta\phi_j(\mathbf{x}) + \dots \quad (2)$$

In this expression,  $N$  is the number of  $e$ -folds of expansion, from a flat slice of spacetime on which the light fields during inflation have values  $\phi_i(\mathbf{x}) = \phi_i + \delta\phi_i(\mathbf{x})$ , and ending on a slice which has uniform energy density  $\rho$ . The initial slice is taken to be a few  $e$ -folds after the relevant scales have left the horizon. The final slice can be any time after  $\zeta$  has settled down to the time-independent value which provides an initial condition for the evolution of perturbations after horizon entry, and is constrained by observation. We use the notation  $N_i \equiv \partial N / \partial \phi_i$  and  $N_{ij} \equiv \partial^2 N / \partial \phi_i \partial \phi_j$ , the derivatives being evaluated with the fields at their unperturbed values  $\phi_i$ .

According to first-order cosmological perturbation theory, the field perturbations  $\delta\phi_i(\mathbf{x})$  are gaussian, with spectrum  $(H/2\pi)^2$  where  $H$  is the Hubble parameter during inflation. (In this paper we ignore the scale dependence of the spectrum.) Using this result, Eq. (2) gives the evolution of  $\zeta$  without any further use of cosmological perturbation theory. The first term is Gaussian, and higher terms are responsible for any non-gaussianity.

The question is whether it is permissible to ignore the non-Gaussianity of  $\delta\phi_i$  which is generated at higher orders in cosmological perturbation theory. In this paper we answer the question in the affirmative, at least for the next order in cosmological perturbation theory and for the three-point correlator of  $\zeta$ . Our starting point is a recent calculation of Seery and Lidsey [12]

## 2. The three point correlator and $f_{\text{NL}}$

Since observation shows that  $\zeta$  is almost Gaussian, Eq. (2) must be dominated by the first term. The spectrum of the curvature perturbation is therefore [5]

$$\mathcal{P}_\zeta \simeq \left(\frac{H}{2\pi}\right)^2 \sum_{i=1}^n N_i^2, \quad (3)$$

The three-point correlator of  $\zeta$ , or its bispectrum defined by  $\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 B_\zeta \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$ , is the lowest order signature of non-Gaussianity. Following Maldacena [11], we define  $f_{\text{NL}}(k_1, k_2, k_3)$  by

$$B_\zeta(k_1, k_2, k_3) = -\frac{6}{5} f_{\text{NL}} [P_\zeta(k_1) P_\zeta(k_2) + \text{cyclic}], \quad (4)$$

where  $P_\zeta(k) = 2\pi^2 \mathcal{P}_\zeta / k^3$ .

At the level of first-order cosmological perturbation theory, the the perturbations in the fields are Gaussian. Then  $f_{\text{NL}}$  is almost scale-independent and given by [3, 4]

$$-\frac{6}{5} f_{\text{NL}} \simeq \frac{\sum_{ij} N_i N_j N_{ij}}{(\sum_i N_i^2)^2} + \mathcal{P}_\zeta \frac{\sum_{ijk} N_{ij} N_{jk} N_{ki}}{(\sum_i N_i^2)^3}. \quad (5)$$

At the level of second-order cosmological perturbation, the field perturbations have non-Gaussianity which is specified entirely by their three-point correlator. This adds to  $f_{\text{NL}}$  the following contribution [3] ‡

$$\Delta f_{\text{NL}} = \frac{\sum_{i,j,k} N_i N_j N_k f_{\text{NL}}^{ijk}(k_1, k_2, k_3)}{(\sum_i N_i^2)^{3/2} \mathcal{P}_\zeta^{1/2}}, \quad (6)$$

where the  $f_{\text{NL}}^{ijk}(k_1, k_2, k_3)$  functions are related to the three point correlation functions of the fields by

$$B_\phi^{ijk}(k_1, k_2, k_3) = - (4\pi^4) \frac{6}{5} f_{\text{NL}}^{ijk}(k_1, k_2, k_3) \left(\frac{H}{2\pi}\right)^3 \frac{\sum_i k_i^3}{\prod_i k_i^3}, \quad (7)$$

$$\langle \delta\phi_{\mathbf{k}_1}^i \delta\phi_{\mathbf{k}_2}^j \delta\phi_{\mathbf{k}_3}^k \rangle = (2\pi)^3 B_\phi^{ijk}(k_1, k_2, k_3) \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \quad (8)$$

The quantity  $f_{\text{NL}}$  will eventually be observable [1] if  $|f_{\text{NL}}| \sim 1$ . It will then be given accurately by Eq. (5), provided that  $|\Delta f_{\text{NL}}| \ll 1$ .

Seery and Lidsey [12] find from second-order cosmological perturbation theory

$$\langle \delta\phi_{\mathbf{k}_1}^i \delta\phi_{\mathbf{k}_2}^j \delta\phi_{\mathbf{k}_3}^k \rangle = (2\pi)^3 \frac{\delta^3(\sum_i \mathbf{k}_i)}{\prod_i k_i^3} \frac{4\pi^4}{M_P^2} \left(\frac{H}{2\pi}\right)^4 \sum_{\sigma'} \frac{\dot{\phi}_i}{2H} \delta_{jk} \mathcal{M}_{123}. \quad (9)$$

‡ Other contributions will be studied in a separate paper, but are expected to be subdominant.

Here the sum  $\sigma'$  is over all the permutations of the indices  $i, j$  and  $k$ , at the same time their respective momenta  $k_1, k_2$  and  $k_3$ , and

$$\mathcal{M}_{123}(k_1, k_2, k_3) = \frac{1}{2} \left( -3 \frac{k_2^2 k_3^2}{k_t} - \frac{k_2^2 k_3^2}{k_t^2} (k_1 + 2k_3) + \frac{1}{2} k_1^3 - k_1 k_2^2 \right), \quad (10)$$

where  $k_t = k_1 + k_2 + k_3$ . Then  $\Delta f_{\text{NL}}$  can be read off from (7), (8) and (9) resulting in

$$-\frac{6}{5} \Delta f_{\text{NL}} = \frac{\sum_i N_i \left( \frac{\phi_i}{2H} \right) \sum_\sigma \mathcal{M}_{123}}{M_P^2 \sum_i N_i^2 \sum_i k_i^3}. \quad (11)$$

The sum over  $\sigma$  denotes the sum over the permutations of the three momenta only.

If the only relevant field perturbation is that of the inflaton in a single-component slow-roll model of inflation, the sum of Eqs. (5) and (11) reproduce [12] the result of Maldacena [11]. In that case  $|f_{\text{NL}}| \ll 1$ , making it too small to observe.

### 3. The maximum of the $\Delta f_{\text{NL}}$ function for constant $\zeta$ .

To maximise  $\Delta f_{\text{NL}}$  with fixed  $\zeta$  we use the Lagrange multipliers method. To do this, first write equation (11) in the following form

$$\frac{6}{5} \Delta f_{\text{NL}} \simeq \frac{\sum_i N_i V^i \mathcal{C}(k_1, k_2, k_3)}{\sum_i N_i^2}, \quad (12)$$

where we have used the slow roll condition  $3H\dot{\phi}^i \simeq -V^i$ , and  $\mathcal{C}$  is a function of the momenta only given by

$$\mathcal{C}(k_1, k_2, k_3) = \frac{1}{6H^2 M_P^2} \frac{\sum_\sigma \mathcal{M}_{123}}{\sum_i k_i^3}. \quad (13)$$

Then, the differential of  $\Delta f_{\text{NL}}$  and the constraint are respectively

$$d(\Delta f_{\text{NL}}) = \frac{\mathcal{C}}{\sum_i N_i^2} \sum_i V^i dN_i, \quad (14)$$

$$\sum_i 2\lambda N^i dN_i = 0, \quad (15)$$

where  $\lambda$  is the Lagrange multiplier corresponding to the unique constraint of the problem,  $\sum_i N_i^2 = \text{constant}$ . Adding terms proportional to  $dN_i$  we find that

$$N_i = -\frac{\mathcal{C}}{2\lambda} \frac{V_i}{\sum_i N_i^2}. \quad (16)$$

To find the value of  $\lambda$  we need to add all the  $N_i^2$ 's. After doing this one gets

$$\lambda = \pm \frac{\mathcal{C} (\sum_i V_i^2)^{1/2}}{2 (\sum_i N_i^2)^{3/2}}, \quad (17)$$

and therefore the extrema of the function corresponds to the values

$$N_i = \pm \left( \sum_j N_j^2 \right)^{1/2} \frac{V_i}{\left( \sum_j V_j^2 \right)^{1/2}} = \pm \frac{\mathcal{P}_\zeta^{1/2}}{(H/2\pi)} \frac{V_i}{\left( \sum_j V_j^2 \right)^{1/2}}. \quad (18)$$

To find the extrema of the non-linear function  $f_{\text{NL}}$  we substitute the value above of  $N_i$  in equation (12), which yields

$$\frac{6}{5}|\Delta f_{\text{NL}}|_{\text{max}} = \frac{1}{6} \frac{|\sum_{\sigma} \mathcal{M}_{123}|}{\sum_i k_i^3} \frac{(H/2\pi)}{\mathcal{P}_{\zeta}^{1/2} M_P^2} \sqrt{\sum_i \left(\frac{V_i}{H^2}\right)^2}. \quad (19)$$

Writing explicitly the terms in the sum  $\sum_{\sigma} \mathcal{M}_{123}/\sum_i k_i^3$  one gets

$$\left| \frac{\sum_{\sigma} \mathcal{M}_{123}(k_1, k_2, k_3)}{\sum_i k_i^3} \right| = \left| \frac{1}{2} - \frac{1}{2} \frac{\sum_{i \neq j} k_i k_j^2}{\sum_i k_i^3} - 4 \frac{\sum_{i > j} \frac{k_i^2 k_j^2}{k_t}}{\sum_i k_i^3} \right| \quad (20)$$

The maximum value of the quantity in brackets is  $11/6$ , achieved for  $k_1 = k_2 = k_3$ .

The slow roll parameter  $\epsilon$  is defined along the steepest descent trajectory of the potential  $V(\phi_i)$ . That is,

$$\epsilon \equiv \frac{M_P^2}{2} \frac{|\vec{\nabla} V|^2}{V^2} = \frac{1}{18M_P^2} \sum_i \left(\frac{V_i}{H^2}\right)^2, \quad (21)$$

Then,  $\Delta f_{\text{NL}}$  can be written in the following form:

$$\frac{6}{5}|\Delta f_{\text{NL}}|_{\text{max}} = \frac{11}{12} \frac{(H/2\pi)}{\mathcal{P}_{\zeta}^{1/2} M_P} \sqrt{2\epsilon}. \quad (22)$$

The power spectrum of gravitational waves,  $\mathcal{P}_G = 8M_P^{-2}(H/2\pi)^2$ , is independent of the number of fields, and therefore we can express the tensor to scalar ratio  $r$  as

$$r = \frac{\mathcal{P}_G}{\mathcal{P}_{\zeta}} = \frac{8(H/2\pi)^2}{M_P^2 \mathcal{P}_{\zeta}}. \quad (23)$$

Introducing  $r$  in equation (22) for  $\Delta f_{\text{NL}}$ , one gets

$$\frac{6}{5}|\Delta f_{\text{NL}}|_{\text{max}} = \frac{11}{24} \sqrt{r\epsilon}. \quad (24)$$

We can use a recent analysis [13] of observations to bound  $r$  and  $\epsilon$ . There is a direct bound  $r < 0.46$ . Also, the bound  $|n - 1| < 0.04$  on the spectral tilt, combined with the prediction [14, 15]  $n - 1 = 2\epsilon + \dots$  gives  $\epsilon \lesssim 0.02$  (barring an accurate cancellation in the last formula). This gives

$$\frac{6}{5}|\Delta f_{\text{NL}}| \lesssim 0.044. \quad (25)$$

We conclude that the three-point correlator of  $\zeta$  can safely be calculated from Eq. (5), if it is big enough to be observable.

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